Markov Chains Mixing Times **December 7, 2022**

Lecture 17: Isoperimetric inequality and Lamplighter Walk

Lecturer: Yuval Peres Week: 9

Isoperimetic Inequality

Theorem 17.115 (Discrete isoperimetric inequality))**.** *Let $A \subset \mathbf{Z}^d$ be a finite set, then*

$$|\partial A| \geq 2d|A|^{\frac{d-1}{d}}.$$

Remark 17.116. Observe that the $2d$ constant in the inequality is the best possible as the example of the d-dimensional cube shows: If $A = [0,n)^d \cap \mathbf{Z}^d$, then $|A| = n^d$ and $|\partial A| = 2dn^{d-1}$.

Theorem 17.117 (Discrete Loomis and Whitney inequality)**.** *For every $1 \leq i \leq d$, define the projection $\mathcal{P}_i : \mathbf{Z}^d \to \mathbf{Z}^{d-1}$ simply as the function dropping the ith coordinate, i.e.,*

$$\mathcal{P}_i\left(x_1, \dots, x_d\right) = \left(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d\right).$$

Theorem 17.115 follows easily from the following lemma.

Lemma 17.118 (Discrete Loomis and Whitney inequality, 1949)**.** *For any finite $A \subset \mathbf{Z}^d$,*

$$|A|^{d-1} \leq \prod_{i=1}^{d} |\mathcal{P}_i(A)|$$

Proof of Theorem 17.115. The important observation is that $|\partial A| \geq 2\sum_{i=1}^d |\mathcal{P}_i A|$. To see this, observe that any vertex in $\mathcal{P}_i(A)$ matches to a straight line in the i th coordinate direction which "stabs" A. Thus, since A is finite, to any vertex in $\mathcal{P}_i(A)$ you can always match two distinct edges in ∂A : the first and last edges on the straight line which intersects A. Using this and the arithmetic-geometric mean inequality we get

$$|A|^{d-1} \leq \prod_{i=1}^{d} |\mathcal{P}_i(A)| \leq \left(\frac{1}{d}\sum_{i=1}^{d} |\mathcal{P}_i(A)|\right)^d \leq \left(\frac{|\partial A|}{2d}\right)^d$$

as required. □

Entropy and conditional entropy

To prove Lemma 17.118, we introduce some notions of entropy.

Definition 17.119 (Entropy and conditional entropy)**.** Let X and Y be random variables that take values $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, respectively. Denote $p(x) := \mathbb{P}[X = x]$ and $p(x,y) := \mathbb{P}[X = x, Y = y]$. Then, the entropy of X is defined to be

$$H(X) = \sum_{i=1}^{n} p(x_i) \log (1/p(x_i))$$

The conditional entropy $H(X \mid Y)$ of X given Y is defined as

$$\begin{aligned} H(X \mid Y) &= H(X,Y) - H(Y) \\ &= \sum_{x_i, y_j} p(x_i, y_j) \log (1/p(x_i, y_j)) - \sum_{y_j} p(y_j) \log (1/p(y_j)) . \end{aligned}$$

Proposition 17.120. *Let us look at some simple properties of entropy,*

(i) If X takes n values, then $H(X) \le \log n$.

(ii) $H(X \mid Y) \le H(X)$ and $H(X \mid Y, Z) \le H(X \mid Z)$.

Proof. (i) Note that $H(X) - \log n = \sum_{i=1}^{n} p(x_i) \log \frac{1}{np(x_i)}$. Plugging in the inequality $\log t \le t - 1$, valid for all $t > 0$, we get

$$H(X) - \log n \le \sum_{i=1}^{n} p(x_i) \left(\frac{1}{np(x_i)} - 1 \right) = 0$$

(ii) Write the difference as $H(X|Y) - H(X) = H(X,Y) - H(Y) - H(X)$.

$$\begin{aligned} H(X|Y) - H(X) &= \sum_{i,j} p(x_i, y_j) \log (1/p(x_i, y_j)) - \sum_{y_j} p(y_j) \log (1/p(y_j)) - \sum_{i=1}^{n} p(x_i) \log (1/p(x_i)) \\ &= \sum_{i,j} p(x_i, y_j) \left[\log (1/p(x_i, y_j)) - \log (1/p(y_j)) - \log (1/p(x_i)) \right] \\ &= \sum_{i,j} p(x_i, y_j) \left[\log \left(\frac{p(x_i)p(y_j)}{p(x_i, y_j)} \right) \right] \\ &\le \sum_{i,j} p(x_i, y_j) \left(\frac{p(x_i)p(y_j)}{p(x_i, y_j)} - 1 \right) \\ &= \sum_{i,j} p(x_i)p(y_j) - p(x_i, y_j) = 0 \end{aligned}$$

□

Theorem 17.121 (Han-Shearer inequality)**.** *Let $X_1, \dots, X_d$ be random variables taking finitely many values, and let $S_1, \dots, S_I \subset \{1, \dots, d\}$ be sets with the property that any $j = 1, \dots, d$ is a member of precisely r of the S_i 's. Then*

$$rH(X_1, \dots, X_d) \le \sum_{i=1}^{I} H(\{X_j : j \in S_i\})$$

Proof. For any i, by definition, we have the telescoping sum

$$H(X_j : j \in S_i) = \sum_{j \in S_i} H(X_j \mid X_u : u \in S_i, u < j) \ge \sum_{j \in S_i} H(X_j \mid X_u : u < j),$$

where the last inequality follows from part (ii) of Proposition 17.120. Sum over i and recall that each j appears in precisely r of the S_i 's to get

$$\begin{aligned}\sum_{i=1}^{I} H(X_j : j \in S_i) &\ge \sum_{i=1}^{I} \sum_{j \in S_i} H(X_j \mid X_u : u < j) \\ &= \sum_{j=1}^{d} rH(X_j \mid X_u : u < j) = rH(X_1, \dots, X_d),\end{aligned}$$

where the last equality follows from the definition of conditional entropy. □

Proof of Lemma 17.118. Take random variables $X_1, \dots, X_d$ such that $(X_1, \dots, X_d)$ is distributed uniformly on A. It is clear that $H(X_1, \dots, X_d) = \log |A|$, and by Proposition 17.120,

$$H(X_1, \dots X_{i-1}, X_{i+1}, \dots, X_d) \le \log |\mathcal{P}_i(A)|.$$

Now use Theorem 17.121 on $X_1, \dots, X_d$ and $S_i = \{1, \dots, d\} - \{i\}$ to find that

$$(d-1) \log |A| \le \sum_{i=1}^{d} \log |\mathcal{P}_i(A)|,$$

as required. □

We now present an argument which gives a lower bound to the isoperimetric profile of a Cayley graph based only on the rate of growth of the graph. This proof is attributed to Coulhon and Saloff-Coste. Denote by $B(n)$ the ball of radius n surrounding the identity and by $V(n)$ the number of vertices in $B(n)$.

Theorem 17.122. *Let $G = (V, E)$ be a Cayley graph of degree d. Let $\phi(\ell) = \inf\{n : V(n) > \ell\}$, then for all finite $K \subset V$ such that $|K| \leq |V|/2$, we have*

$$\frac{|\partial K|}{|K|} \geq \frac{1}{2\phi(2|K|)}$$

Proof. Let $n = \phi(2|K|)$ and take the ball such that $V(n) \geq 2|K|$. Fix $x \in K$, then a uniform random $g \in B(n)$ has probability at least $1/2$ to have $gx \notin K$. So the expected value of $|\{x \in K : gx \notin K\}|$ is at least $|K|/2$, hence there is some $g \in B(n)$ that achieves this. Now notice that if s is a generator, the number of vertices that leave K due to s acting on them is at most $|\partial K|$, i.e., $|\{x \in K : sx \notin K\}| \leq |\partial K|$. By the same token, if g is at distance at most n from the origin, then $|\{x \in K : gx \notin K\}| \leq n|\partial K|$, so we get

$$|K|/2 \leq n|\partial K|,$$

which yields the result. □

Example 17.123. In $\mathbb{Z}^d$ or in $\mathbb{Z}_m^d$ we have $V(n) \asymp c_d n^d$. Hence, $\phi(l) \asymp \tilde{c}_d l^{1/d}$ which further shows that,

$$\frac{|\ \partial k\ |}{|k|} \geq \tilde{c}\left(\frac{2}{k}\right)^{-1/d}$$
$$|\partial k| \geq c^*|k|^{\frac{d-1}{d}}.$$

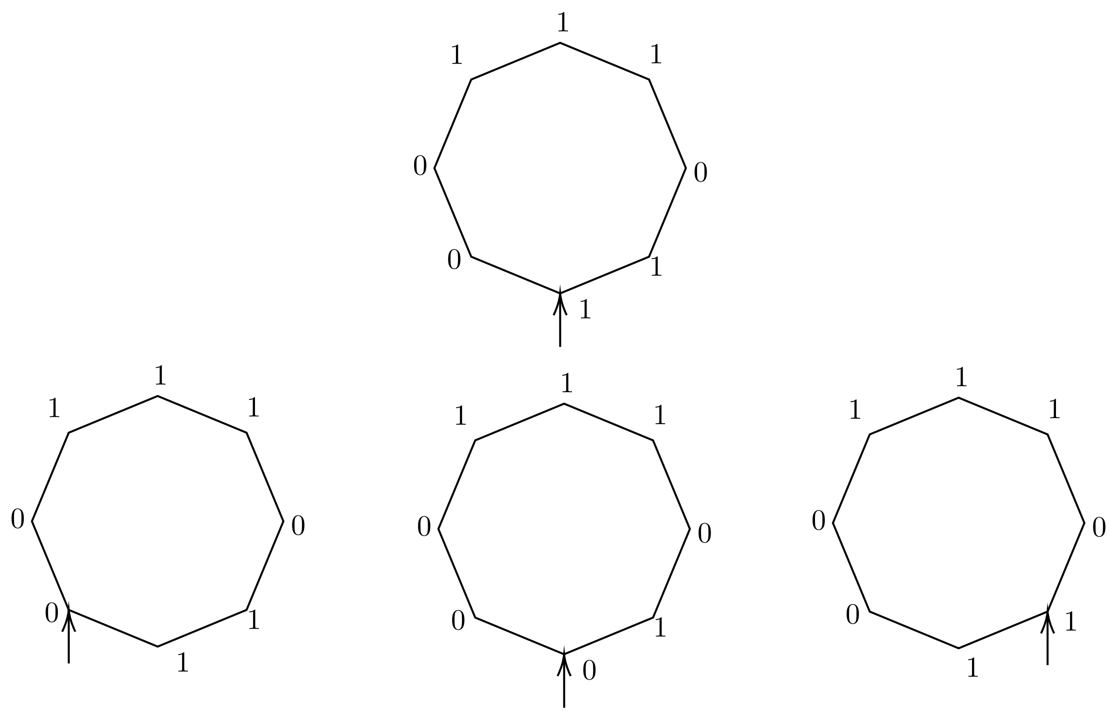

Figure 17.2: The top diagram shows the configuration of the lamps and position of the lamplighter in a state. The rest three diagrams show the three possible states in the next step.

Lamplighter Walks

Let $G = (V, E)$ be a graph, then the wreath product $G^{\diamond}$ is defined as the graph with vertex set, $V^{\diamond} = \{0,1\}^V \times V$ and (f, u) (g, v) iff v and u are identical or adjacent, and f and g agree on every vertex other than v and u.

Example 17.124 (Lamplighter walk on n-cycle)**.** Consider n lamps being placed on the graph. Now a walker is placed in one vertex who can either move to its neighbors or switch the light at that vertex as shown in Figure 17.2. This is essentially a simple random walk on the lamplighter graph. For an n-cyccle the lamplighter graph is of size $n2^n$.

Remark 17.125. The most importance feature of this model is that the mixing time is is bounded both above and below by the cover time asymptotically.